OUTP-96-43 P
TPI-MINN-96/11-T
February 1, 2008
hep-th/9608031



# THREE-DIMENSIONAL DESCRIPTION OF THE $\Phi_{1,3}$ DEFORMATION OF MINIMAL MODELS

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ABSTRACT

We discuss the 2 + 1 dimensional description of the $\Phi_{1,3}$ deformation of the minimal model $M_p$ leading to a transition $M_p \to M_{p-1}$. The deformation can be considered as an addition of the charged matter to the Chern-Simons theory describing a minimal model. The $N = 1$ superconformal case is also considered.

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It is known that two-dimensional conformal field theories ($2D$ CFT) [1], [2] can be described in three-dimensional terms by using an amusing connection [3] between a $2+1$-dimensional topological Chern-Simons (CS) theory with an action

$$kS_{CS}\{A\} = \frac{k}{4\pi}\int_{\mathcal{M}} Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{1}$$

defined on a three-dimensional manifold $M$ with a boundary $\Sigma = \partial M$ and a Wess-Zumino-Novikov-Witten (WZNW) model on a two-dimensional boundary $\Sigma$. It is also known that two-dimensional conformal field theories can be considered as fixed points (infrared or ultraviolet) of the renormalization group (RG) flows in the space of all $2D$ renormalizable quantum field theories (QFT). In this approach, which was initiated in [4], [5], a conformal field theory with an action $S_{CFT}$ is deformed by some operators $V_i$ with scaling dimensions $d_i = \Delta_i + \bar{\Delta}_i$ and the action is

$$S[\mu] = S_{CFT} + \sum_i \lambda^i[\mu]\int d^2\xi\ V_i(\xi) \tag{2}$$

where coupling constants $\lambda^i$ depend on a scale $\mu$ as well as the whole action (2). In the case of a relevant deformation $d_i < 2$ the coupling constants vanish in the ultraviolet (UV) limit and one can recover conformal field theory $S_{CFT}$ as an UV fixed point. In the case of an irrelevant deformation $d_i > 2$ the coupling constants vanish in the infrared (IR) limit and one gets conformal field theory $S_{CFT}$ as an infrared fixed point of the renormalization group flow.

If the renormalization group exibits topologically nontrivial behaviour, i.e. has another fixed point in the vicinity of the original one, there is a RG flow connecting *two* different conformal field theories. The toy model of this phenomenon is the famous RG flow $M_k \to M_{k-1},\ k = 1, 2, 3, 4, \ldots$ [4] [5] where the action (2) interpolates between the UV fixed point $M_k$ and the IR fixed point $M_{k-1}$ and the deformation operator $V$ is a $\Phi_{1,3}$ field with anomalous dimensions $\Delta_{1,3} = \bar{\Delta}_{1,3} = 1 - 2/(k+3)$ in a vicinity of the UV fixed point $M_k$. Let us note that we are using here nonstandard notation, the standard one for minimal models is $M_p,\ \ p = k + 2 = 3, 4, \ldots.$

Minimal models can be described by the $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$ GKO coset construction [6] and a corresponding Chern-Simons description is given by three $SU(2)$ gauge fields $A$, $B$ and $C$ with action [7]

$$kS_{CS}\{A\} + S_{CS}\{B\} - (k+1)S_{CS}\{C\} \tag{3}$$

We shall use obvious notation $[k, 1; k+1] = [k, 1, -(k+1)]$ for the triplets of CS coefficients. For the $M_k \rightarrow M_{k-1}$ flow one has *two different* Chern-Simons theories corresponding to the UV and IR fixed points of the flow. One can ask immediately the following questions:

- *What is necessary to add to a $2+1$-dimensional topological CS theory to be able to get a three-dimensional description of a deformed conformal field theory ?*
- *How will this new ingredient change the original CS theory corresponding to the $M_k$ model into a new one which must describe the $M_{k-1}$ model in the infrared limit?*

The answer to the first question (in general) is known [8] - we have to add charged matter, so that the three-dimensional theory is not topological anymore - there are propagating degrees of freedom in the bulk. But it was unclear how to add a charged matter in such a way that in the infrared limit the new CS theory will arise. The aim of this letter is to answer the second question and to discuss the three-dimensional description of the two-dimensional RG flow $M_k \rightarrow M_{k-1}$ as well as it SUSY generalization.

Let us remind how a charged matter added in the bulk will induce a deformed $2D$ CFT on the boundary (for details see [8]). The statistical sum of a deformed conformal field theory ( in a critical string theory it gives a generating function in an external field) is determined by the action (2) and in case of a single-charge deformation is given by

$$Z = \int D\Phi(\xi) exp[-S_{CFT} + \lambda \int d^2\xi \; V(\xi, \bar{\xi})] =$$
$$\sum_n \frac{\lambda^n}{n!} \int d^2\xi_1 \cdots \int d^2\xi_n \langle V(\xi_1, \bar{\xi}_1) \cdots V(\xi_n, \bar{\xi}_n) \rangle, \tag{4}$$

where the brackets $\langle V(\xi_1, \bar{\xi}_1) \cdots V(\xi_n, \bar{\xi}_n) \rangle$ are $n$-point correlation functions in an unperturbed CFT which can be represented as products of left and right conformal blocks $\langle V(\xi_1, \bar{\xi}_1) \cdots V(\xi_n, \bar{\xi}_n) \rangle = \langle V_L(\xi_1) \cdots V_L(\xi_n) \rangle \; \langle V_R(\bar{\xi}_1) \cdots V_R(\bar{\xi}_n) \rangle$ where $V_L(\xi)$ and $V_R(\bar{\xi})$ are holomorphic and antiholomorphic chiral vertex operators corresponding to the left-right symmetric operator $V(\xi, \bar{\xi})$. To obtain a three-dimensional picture we have to consider a membrane with topology $M = \Sigma \times I$ where the two boundaries $\Sigma_L$ and $\Sigma_R$ are connected by an interval $I$ of a length $\beta$. A gauge theory with a Chern-Simons term in the bulk will induce left and right sectors of a $2D$ CFT (actually it depends on boundary conditions for gauge fields, for more details see [9]) and an insertion of the vertex operator $V(\xi, \bar{\xi})$ on a world-sheet $\Sigma$ is equivalent to insertions of chiral vertex operators $V_L(\xi)$ and $V_R(\bar{\xi})$ on left and right world-sheets $\Sigma_L$ and $\Sigma_R$ respectively with the coordinates on both

world-sheets identified in the end. This is induced by an open Wilson line

$$W_{\{R_k\}}(C_{\xi_L,\xi_R}) = \prod_k \; Tr_{R_k} \; \exp\left(i \int A_\mu^{(k)} dx^\mu\right) \tag{5}$$

along the path $C_{\xi_L,\xi_R}$ with end points $\xi_L$ and $\xi_R$ on left $\Sigma_L$ and right $\Sigma_R$ world-sheets respectively. [3], [7]-[10]. The insertion of this Wilson line in the bulk gives the phase factor of a propagating charged particle where charges with respect to gauge fields $A^{(k)}$ are given by representations $R_k$ (this set of quantum numbers depends on the type of vertex operator under consideration). The quantum particle propagates from left to right world-sheets and a gas of these open Wilson lines describes a charged matter in the bulk. The third dimension along the interval $I$ plays the role of an imaginary time and the parameter $\beta$ (internal size of a membrane) can be interpreted as an inverse temperature $\beta = T^{-1}$. In this way the connection between a charged $2+1$-dimensional matter at a temperature $T$ and a deformed two-dimensional conformal field theory is established.

It is easy to see that the fugacity $\lambda$ depends on the temperarture $T = 1/\beta$. The Wilson line (5) is a phase factor in the path integral describing the propagation of the quantum particle with mass $m$ (let us for simplicity consider the simplest case of scalar particle, in other cases the conclusion will be the same) from $\Sigma_L$ to $\Sigma_R$

$$G(\xi_L,\xi_R) = \int_0^\infty d\tau \exp\left[-\frac{1}{2}m^2\tau\right] \int_{x(0)=\xi_L}^{x(\tau)=\xi_R} \mathcal{D}x(t) \exp\left[-\frac{1}{2}\int_0^\tau dt\; \dot{x}^\mu \dot{x}_\mu\right] W_{\{R_k\}}(C_{\xi_L,\xi_R}) \tag{6}$$

where $x^\mu(t)$ is a three-dimensional coordinate along a quantum path and $\tau$ is the proper time. The classical path is a straight line $x^\mu(t) = \delta^{\mu,3}\beta\; t/\tau$ and an extremal value of the proper time is $\tau = \beta/m$ from which one gets the leading (classical) contribution $\exp[-\beta\; m] W_{\{R_k\}}(C_{\xi_L,\xi_R})$ and each Wilson line is accompanied by a fugacity factor $\lambda \sim \exp[-\beta\; m]$. In the low-temperature limit $(T/m) \to 0$, $(m\beta) \to \infty$ the fugacity, i.e. deformation parameter $\lambda$ disappears and we have a conformal field theory. It is necessary to have in mind that the same charge matter can, in principle, renormalize the parameters of the Chern -Simons terms. The problem is that besides the Wilson lines connecting left and right world-sheets (using our analogy with finite temperature one can see that they are nothing but Polyakov's lines) there are ordinary closed loops in the bulk which do not touch the boundary. These last ones do not induce any vertex operators insertions and are not supressed in the low $T$ limit. They describe production and annihilation of virtual pairs and any vacuum loops will renormalize parameters of the gauge theory. If the charged matter is $P$-odd, then these loops give contributions to the total Chern-Simons coefficients which become $T$- dependent. Thus we see that the bulk parameters will also

experience some kind of flow from small $T$ to large $T$. At the same time the fugacity $\lambda(T)$ starts to increase and as a result the coupling constant of the induced two-dimensional theory starts to increase. In the limit of infinite $T$ or zero $\beta$ we can reach (if we have matter with suitable charges) a new set of Chern-Simons coefficients describing another conformal field theory. In such a way we obtain the three-dimensional (membrane or bulk) description of a two-dimensional RG flow. Let us note that a matter contribution to the initial set of the CS coefficients is calculated at zero (or very low) temperature $T << m$. The same contribution to the final set (corresponding to the IR fixed point) must be calculated in the high-temperature limit $T >> m$, in which case the matter contribution will be proportional to $\tanh(m/T) \to 0$ and can be neglected.

Let us apply now these ideas to $M_k \to M_{k-1}$ flow. It seems that we have to demonstrate that matter contribution must change Chern-Simons coefficients in (3) from $(k-1, 1; k)$ into $(k, 1; k+1)$. To see if this is possible the quantum numbers of the $\Phi_{1,3}$ operator must be identified first of all. As was demonstrated by Goddrad, Kent and Olive [6] representations of the affine Kac-Moody algebra $\widehat{SU(2)}_k \times \widehat{SU(2)}_1$ can be decomposed with respect to $\widehat{SU(2)}_{k+1} \times V(c)$, where $V(c)$ denotes the Virasoro algebra of the minimal model $M_k$ with central charge

$$c_{M_k} = c_{SU(2)_k} + c_{SU(2)_1} - c_{SU(2)_{k+1}} = 1 - \frac{6}{(k+2)(k+3)} \tag{7}$$

Highest weight irreducible unitary representations of $\widehat{SU(2)}_k$ are labelled by $(k, l)$ and called level $k$, spin $l$ representations, where $l$ is the spin ($SU(2)$ charge) of the corresponding primary field and the restriction $0 \leq 2l \leq k$ must apply. The product of two representations $(k, l)$ and $(1, \epsilon)$ decomposes into the direct sum

$$(k, l) \times (1, \epsilon) = \bigoplus_{l'} \left(k+1, \frac{1}{2}(q-1)\right) \times (c, \Delta_{2l+1, 2l'+1}(c)) \tag{8}$$

where $c$ is given by (7) and

$$\Delta_{p,q}(c) = \frac{[(k+3)p - (k+2)q]^2 - 1}{4(k+2)(k+3)} \tag{9}$$

are anomalous dimensions of the primary fields with respect to the Virasoro algebra $V(c)$. The sum in (8) is taken over $l'$ such that $2(l - l')$ is even or odd, depending on whether $\epsilon = 0$ or $1/2$, and $1 \leq q = 2l' + 1 \leq k + 2$. The decomposition (8) implies the following relations between characters

$$\chi_{k,l}(q, \theta)\ \chi_{1,\epsilon}(q, \theta) = \sum_{l'} \chi_{k+1,l'}(q, \theta) \chi^V_{c, \Delta_{2l+1, 2l'+1}}(q) \tag{10}$$

where

$$\chi_{k,l}(q,\theta) = Tr\left(q^{L_0} exp(i\theta T^3)\right) = q^{l(l+1)/(k+2)} \times$$

$$\frac{\sum_{n\in Z} q^{n^2(k+2)+n(2l+1)} \{\exp(i(l+n(k+2))\theta) - \exp(-i(l+1+n(k+2))\theta)\}}{\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{i\theta})(1-q^{n-1}e^{-i\theta})} \tag{11}$$

and

$$\begin{aligned} \chi^V_{c,\Delta_{p,q}}(q) = Tr_{\Phi_\Delta}\left(q^{L_0}\right) &= \sum_{n\in Z}\left\{q^{\alpha(n)} - q^{\beta(n)}\right\}\prod_{n=1}^{\infty}(1-q^n)^{-1} \\ \alpha(n) &= \frac{[(k+3)p-(k+2)q+2n(k+2)(k+3)]^2-1}{4(k+2)(k+3)}; \\ \beta(n) &= \frac{[(k+3)p+(k+2)q+2n(k+2)(k+3)]^2-1}{4(k+2)(k+3)} \end{aligned} \tag{12}$$

are characters for Kac-Moody [11] and Virasoro [12] algebras respectively.

Thus to get the field $\Phi_{1,3}$ we must take $l=0$, $\epsilon=0$ and $l'=1$. Let us also mention that the field $\Phi_{3,1}$ into which the first one must flow when approaching the IR fixed point $M_{k-1}$, must have $l=1$, $\epsilon=0$ and $l'=0$. In spite of the fact that $\epsilon=0$ it is wrong that the $\Phi_{1,3}$ field has no charge with respect to $SU(2)_1$. It is easy to see using a very simple expression for the Kac-Moody character at level 1

$$\chi_{1,\epsilon}(q,\theta) = \sum_{m\in Z+\epsilon} q^{m^2} e^{im\theta}\prod_{n=1}^{\infty}(1-q^n)^{-1} \tag{13}$$

and (10), (11) and (13) that for $l'=l+n$ ($n$ is an integer for $\epsilon=0$ and a half-integer for $\epsilon=1/2$) the leading terms from $\chi_{1,\epsilon}$ which contribute to the $l'$ term in the sum (10) are $q^{n^2}e^{\pm in\theta}$. This means that $\Phi_{3,1}$ is a descendant of a unity operator with respect to $SU(2)_1$ with a spin $n^2$. This also can be seen from the following representation of the anomalous dimension (9)

$$\begin{aligned} \Delta_{2l+1,2l'+1} = \frac{[(k+3)(2l+1)-(k+2)(2l'+1)]^2-1}{4(k+2)(k+3)} = \\ n^2 + \frac{l(l+1)}{k+2} - \frac{l'(l'+1)}{k+3}; \quad n = l'-l \end{aligned} \tag{14}$$

which makes absolutely clear what part of the anomalous dimension (14) came from each of the three $SU(2)$ sectors. This representation has a straightforward three-dimensional interpretation. It is known (see [13] and references therein) that anomalous dimensions

of $2D$ conformal fields can be obtained from the Aharonov-Bohm part of $2+1$ scattering amplitudes for corresponding dynamical matter fields. From the formula (14) we see that the total anomalous dimension (proportional to the total amplitude) is the sum of three independent contributions from $SU(2)_1$, $SU(2)_k$ and $SU(2)_{k+1}$. The last one is negative, because of the negative sign of the last term in (3).

The obtained quantum numbers for $\Phi_{1,3}$ operator seem very strange, because the corresponding matter field must be charged with respect to $B$ and $C$ fields in (3) but not with respect to the $A$ field. There is no way the first coefficient $k$ will be influenced by presence of this matter field and at first sight it seems impossible to get what we have aimed for. There is a loophole, however. Let us give first a heuristic argument as to what we would like the renormalizations to be. The fact that there is a charge with respect to $SU(2)_1$ makes it possible to renormalize the coefficient 1 also. This second "negative" result actually means that if we shall flip the sign for $SU(2)_1$, i.e. transform 1 into $1-2=-1$ for $B$ field and $k$ into $k-2$ for $C$ field (leaving coefficient $k$ in front of CS action for $A$ field intact) we shall transform $[k,1;k+1]=[k,1,-(k+1)]$ into $[k;1,k-1]=[k,-1,-(k-1)]$ which means that we practically get what we need exept the overall sign. By making a parity transformation (changing the orientation of the 3-manifold $M$) we can change the overall sign of all the CS terms (1). As a result we have the following transformation:

$$
\begin{aligned}
&kS_{CS}\{A\}+S_{CS}\{B\}-(k+1)S_{CS}\{C\} \overset{\text{Renorm.}}{\longrightarrow} kS_{CS}\{A\}-S_{CS}\{B\}-(k-1)S_{CS}\{C\} \\
&\overset{\text{Parity}}{\longrightarrow} (k-1)S_{CS}\{\tilde{A}\}+S_{CS}\{\tilde{B}\}-kS_{CS}\{\tilde{C}\}; \quad (A\ ,B\ ,C)\rightarrow(\tilde{C},\ \tilde{B},\ \tilde{A})
\end{aligned}
\tag{15}
$$

Before demonstrating that precisely this renormalization takes place let us see that the suggested picture of the sign flip of the $B$ field which leads to an effective exchange of roles between the $A$ and $C$ fields is consistent with known renormalizations of the fields $\Phi_{n,m}$ in a perturbed theory $M_{k,k-1}=M_k\rightarrow M_{k-1}$ [4]. To first order the mixing between operators is determined by the fusion rule

$$
\Phi_{n,m}\times\Phi_{1,3}=[\Phi_{n,m}]+[\Phi_{n,m-2}]+[\Phi_{n,m+2}] \tag{16}
$$

where the square brackets denote the contribution of the corresponding operator and of its descendants. This rule is a consequence of the fact that $l=0$ and $l'=1$ for $\Phi_{1,3}$ and as a result the first index is unchanged and for the second one we use the usual rule of the addition of two spins $l'$ and 1 which leads to $l'$ and $l'\pm1$, i.e. in terms of $m=2l'+1$ it leads to $m$ and $m\pm2$. It is also known that only operators with close dimensions are effectively

mixed and then one can conclude [4] that the operator $\Phi_{n,n}$ does not mix with other fields and is the same in both CFT - $M_k$ and $M_{k-1}$. This is in perfect agreement with the fact that this operator corresponds to the matter field with equal charges with respect to $A$ and $C$ fields, so after exchange it will be the same operator. If we consider the pair $\Phi_{n,n\pm1}$ near the UV fixed point $M_k$ it will transform along the flow into the pair $\Phi_{n\pm1,n}$. Again this is nothing but an effective exchange of the $A$ and $C$ fields. Our last example is a triple of fields $\Phi_{n,n\pm2}$ and $\partial_z\partial_{\bar{z}}\Phi_{n,n}$ which transforms into another triple in the infrared - $\Phi_{n\pm2,n}$ and $\partial_z\partial_{\bar{z}}\Phi_{n,n}$. One can study in an analogous manner the renormalizations of the other fields $\Phi_{n,m}$ - and all the time we'll see that it is nothing but the exchange $n \leftrightarrow m$, i.e. nothing but flip of the sign in the $SU(2)_1$ sector.

Now let us demonstrate that the renormalization (15) takes place indeed. It is clear that the only matter fields which can contribute to the parity violating CS terms are fermions or topologically massive gauge bosons. The operator $\Phi_{1,3}$ has a conformal dimension $\Delta_{1,3} = 1 - 2/(k+3)$ (the same for $\bar{\Delta}$) which can be written in the form

$$\Delta_{1,3} = -\frac{1}{2} + 1 + \left(\frac{1}{2} - \frac{2}{k+3}\right) \tag{17}$$

more convenient for our purposes. We also have to remind that conformal dimension defines the spin of the matter field. So equation (17) tells us that our matter field is a combination of non-interacting fermion (the first factor $-1/2$), a vector (the second factor 1) and a fermion in an adjoint representation of $SU(2)_{k+1}$. It is known (see [13] and references therein) that a fermion in a representation $R$ induces the Chern-Simons term $\mathrm{sgn}(m)T(R)S_{CS}$ with the sign depending on the fermion mass $m$ and $T(R)$ defined as $Tr_R T^a T^b = T(R)\delta^{ab}$. For the adjoint representation of the $SU(N)$ group $T(G) = N$ and for $SU(2)$ one gets the shift $k \to k + 2\mathrm{sgn}(m)$. As it has been discussed earlier the $M_k$ model is a UV fixed point where the matter contribution to the total Chern-Simons coefficients must be taken into account. Adjusting the mass $m$ to be negative we can get $-(k+1)$ in front of $S_{CS}\{C\}$ in (3) as $-(k-1)-2$, i.e. without matter the bare CS coefficient was $-(k-1)$ - precisely what we need !

Let us talk about $B$ field corresponding to the $SU(2)_1$ factor. We found that the matter field interacting with $B$ field must have spin one, i.e., it is a vector field $V$ itself and to get a P-odd vertex one can write the lagrangian

$$S_V = \int_{\mathcal{M}} \epsilon^{\mu\nu\lambda} Tr V_\mu \left(\partial_\nu + B_\nu\right) V_\lambda = \int_{\mathcal{M}} \epsilon^{\mu\nu\lambda} Tr V_\mu D_\nu(B) V_\lambda \tag{18}$$

and the induced action after integrating over $V$ will be given by $\ln\ detD(B)$. The same operator appears in the CS action (1) when we split the field $A = B + V$, where $B$ is a

background classical field and $V$ decsribes quantum fluctuations. It is known [3] that after proper regularization one can obtain from this determinant the correction to the classical CS action leading to the shift $k \to k + T(G)\,\mathrm{sgn}(k)$. This shift is obtained using a regularization equivalent to adding a $TrF^2$ term to the Chern-Simons action which transforms topological Chern-Simons theory into topologically massive gauge theory (TMGT) [14]. This regularization is not unique and one can choose another regularization and get a shift $k \to k - T(G)\mathrm{sgn}(k)$ (see [15] for detailed discussion). Actually the sign is dictated by the sign of the mass of the massive vector boson propagating inside the loop. In TMGT this sign is given by the sign of $k$, but in our case the sign of the $V$ particles mass is in our hands. Choosing it opposite to the sign of initial $k$ one gets the shift $k \to k - 2\mathrm{sgn}(k)$. Let us now take $k = -1$ before integration over matter field. After the integration over the matter fields we get the new $k = -1 + 2 = 1$. This is the sign flip of the $SU(2)_1$ factor which was the most crucial element in our construction. Let us also mention that the two different choices of the mass sign for the $V$ field correspond to either $SU(2) \times SU(2)$ or $SO(1,3)$ symmetry of the total action for the $B$ and $V$ fields. In this construction $B$ plays the role of the vector field (rotations) and $V$ plays the role of the axial field (boosts). The choice we make here corresponds to a $SO(1,3)$ symmetry.

As a result we have demonstrated that the transformation (15) is induced by the three-dimensional matter corresponding to the $\Phi_{1,3}$ field. One can repeat the same analysis using $\Phi_{3,1}$ field and demonstrate how in this case the transformation inverse to (15) takes place. The anomalous dimension of the $\Phi_{3,1}$ operator in the $M_{k-1}$ model can be written by analogy with (17) as

$$\Delta_{3,1} = -\left(\frac{1}{2} - \frac{2}{k+1}\right) + 1 - \frac{1}{2} \tag{19}$$

Now we have the adjoint fermion with respect to the first $SU(2)_{k-1}$ group corresponding to the field $\tilde{A} = C$. It is clear also that the sign of the fermion mass now is opposite (because of the parity transformation in (15)). So the fermion contribution will be $+2$ and matter contribution will transform $k-1$ into $k+1$ - and at the same time we shall get the same sign flip for the $B$ field, i.e. it describes the transition from $M_{k-1}$ to $M_k$.

Let us briefly discuss how this construction can be generalized to describe the RG flows between minimal $N = 1$ superconformal models [16] $SM_k$. These models have central charge (note that the central charge of SUSY $SU(2)_k$ WZNW model is $c_{SSU(2)_k} = 3/2 + 3(k-2)/k$ and $k \geq 2$)

$$c_{SM_k} = c_{SSU(2)_k} + c_{SSU(2)_2} - c_{SSU(2)_{k+2}} = \frac{3}{2}\left(1 - \frac{8}{k(k+2)}\right) \tag{20}$$

and can be described by the $SU(2)_k \times SU(2)_2/SU(2)_{k+2}$ coset construction [6]. The dimensions of the primary superfields are given by

$$\Delta_{2l+1,2l'+1} = \frac{[(k+2)(2l+1)-k(2l'+1)]^2-4}{8k(k+2)} + \frac{\epsilon}{8} =$$
$$\frac{n^2}{2} + \frac{l(l+1)}{k} - \frac{l'(l'+1)}{k+2} + \frac{\epsilon}{8}; \quad n = l' - l \tag{21}$$

where for Neveu-Schwartz (NS) superfields $n$ is an integer and $\epsilon = 0$, and for Ramond (R) fields $n$ is a half-integer and $\epsilon = 1/2$. A primary NS superfield $\mathbf{\Phi}_{p,q}(\xi,\theta,\bar{\theta}) = \Phi_{p,q}(\xi) + \theta\Psi_{p,q}(\xi) + \bar{\theta}\bar{\Psi}_{p,q}(\xi) + \tilde{\Phi}_{p,q}(\xi)$ has two boson comnponents $\Phi_{p,q}$ and $\tilde{\Phi}_{p,q}$ with dimensions $(\Delta_{p,q}, \bar{\Delta}_{p,q})$ and $(\Delta_{p,q} + 1/2, \bar{\Delta}_{p,q} + 1/2)$. It is the field $\tilde{\Phi}_{1,3}$ which has dimension $\Delta_{1,3} + 1/2 = 1 - 2/(k+2)$ near 1. The RG flow corresponding to the $\tilde{\Phi}_{1,3}$ deformations was studied in [17], [18]. It was shown in [17] that this flow describes the transition $SM_k \rightarrow SM_{k-2}$, and in [18] this result was confirmed using the Landau-Ginzburg description of the superconformal minimal models. The fact that $\Delta\ k = 0(\mathrm{mod}\ 2)$ is due to the fact that the supersymmetry is not broken by this deformation. The Witten index $Tr(-1)^F$ then must be the same for both UV and IR fixed points and because it equals to $(-1)^k$ for $SM_k$ one must conclude that $k \rightarrow k-2$ is the minimal possible change of $k$. The renormalization of the superfields $\mathbf{\Phi}_{p,q}$ along the RG trajectory $SM_k \rightarrow SM_{k-2}$ was found to be of the same type as in the minimal model case $\mathbf{\Phi}^{(k)}_{p,q} \rightarrow \mathbf{\Phi}^{(k-2)}_{q,p}$.

The three-dimensional description of the $SM_k$ model is given by the $N = 1$ SUSY CS theory with an action

$$kS_{SUSY\ CS}\{A\} + 2S_{SUSY\ CS}\{B\} - (k+2)S_{SUSY\ CS}\{C\} \tag{22}$$

and it is quite clear now that to describe $SM_k \rightarrow SM_{k-2}$ transition we must do the following: $2 \rightarrow 2-4 = -2$ in the second term and $k+2 \rightarrow k+2-4 = k-2$ in the third one. After making a parity transformation we shall exchange the roles of $A$ and $B$ fields and finally get the three-dimensional description of $SM_{k-2}$. The only difference in comparison with the non-SUSY case is that we must subtract 4 and not 2. But this is due to the fact that we add a supersymmetric matter now. So we have to add supermultiplets of fermions and topologically massive vector bosons - and each of them, as it has been shown, contributes 2 to the renormalization of the corresponding CS coefficient, so total contribution will be 4 in both cases. Thus we see that our three-dimensional picture is valid in $N = 1$ superconformal case also.

In conclusion let us discuss several important questions which have to be considered next. First of all it will be interesting to study the $N = 2$ case which is important in

the (super)string theory, especially for description of transitions between different Calabi-Yau manifolds and RG flows on moduli spaces. Let us note that because of the sign flip nature of the three-dimensional transition we have at some scale the Chern-Simons theory of the type $[k, 0; k]$ which corresponds to the coset $(SU(2)_k/SU(2)_k) \times SU(2)_0$. It is interesting to know if the appearence of the topological CFT $G/G$ as well as $SU(2)_0$ will be a generic feature or not. This may be important for the membrane interpretation of the conifold transition [19]. Another important problem is the three-dimensional analog of the Zamolodchikov $c$-function [4]. At the fixed point $c$ equals the corresponding central charge and its three-dimensional interpretation [20], [10], [21] is the gravitational Chern-Simons coefficient in the induced topologically massive gravity [14]. However it is absolutely unclear what is the three-dimensional description of the whole $c$-function and this question as well as many others definitely deserve to be further explored.

**Acknowledgments.**

I would like to thank J.Cardy for very useful discussions and suggestions, particularly for suggesting to check the renormalization of the $\Phi_{n,m}$ fields in the three-dimensional approach and R. Szabo for very stimulating and important discussions. I benefited a lot from very interesting discussions with L. Cooper, A. Lewis, N. Mavromatos, G. Ross, M. Shifman, A. Tsvelik, A. Vainshtein and J. Wheater. This work was partly supported by PPARC travel grant and TPI, University of Minnesota.

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